1. Κανονικοποιείστε τις συναρτήσεις και δείξτε ποιες μπορεί να αποτελέσουν κυματοσυναρτήσεις.

α) e^{-x^2} $\quad (-\infty, \infty)$ $\quad \int_{-\infty}^{\infty} e^{-x^2} e^{-x^2} dx$

$$= \int_{-\infty}^{\infty} e^{-2x^2} dx = \sqrt{\frac{\pi}{2}} \Rightarrow$$

$$\psi(x) = \frac{1}{\sqrt{\sqrt{\pi/2}}} e^{-x^2} = \left(\frac{\pi}{2}\right)^{-1/4} e^{-x^2}$$

ΟΚ, για κυματοσυνάρτηση

(β) $e^{i\theta}$, $(0, 2\pi)$: $\int_0^{2\pi} e^{-i\theta} e^{i\theta}\, d\theta = \int_0^{2\pi} d\theta = 2\pi$

$\psi(\theta) = \frac{1}{\sqrt{2\pi}} e^{i\theta}$, οκ για ιδιοσυνάρτηση

(γ) e^{x}, $(0, \infty)$: $\int_0^{\infty} e^{x} e^{x}\, dx = \int_0^{\infty} e^{2x}\, dx = \frac{1}{2} e^{2x}\Big|_0^{\infty} \longrightarrow \infty$

Το ολοκλήρωμα $\to \infty$ επομένως δεν κανονικοποιείται

ΔΕΝ μπορεί να είναι ιδιοσυνάρτηση

б) xe^{-x} $[0, \infty)$

$$\int_0^\infty xe^{-x}\, xe^{-x}\, dx = \int_0^\infty x^2 e^{-2x}\, dx$$

$$\boxed{\int_0^\infty x^n e^{-x}\, dx = n!}$$

$2x = u \Rightarrow 2dx = du$

$$\int_0^\infty x^2 e^{-2x}\, dx = \int_0^\infty \left(\frac{u}{2}\right)^2 e^{-u} \frac{du}{2} = \frac{1}{8}\int_0^\infty u^2 e^{-u}\, du = \frac{2!}{8} = \frac{1}{4}$$

$$\psi(x) = \frac{1}{\sqrt{4}} x e^{-x} = \frac{1}{2} x e^{-x}$$

ok

2. Ποιες από τις συναρτήσεις είναι εν δυνάμει κυματοσυναρτήσεις.

(α) $\frac{1}{x}$ $(0, \infty)$ $\int_0^\infty \frac{1}{x}\frac{1}{x}\,dx = \int_0^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_0^\infty \longrightarrow \infty$

όχι, διότι δεν κανονικοποιείται

(β) $\frac{1}{1-x^2}$ $(-1, 1)$, $\psi(1) = \frac{1}{1-1^2} = \frac{1}{0} \longrightarrow \infty$

όχι, διότι δεν είναι πεπερασμένη

(γ) $e^{-x}\cos(x)$, $(0, \infty)$

$-1 \leq \cos(x) \leq 1$ ψ(x) πεπερασμένη

$0 \leq e^{-x} \leq 1$

$(x \to \infty)$ $(x = 0)$

$$\int_0^{\infty} e^{-x}\cos(x)\, e^{-x}\cos(x)\, dx = \int_0^{\infty} \cos^2(x)\, e^{-2x}\, dx$$

Η εκτίμηση του ολοκληρώματος είναι πρόβλημα

η απάντηση είναι $3/8$ και επομένως

$$\psi(x) = \frac{1}{\sqrt{3/8}} \cos(x)\, e^{-x} = \sqrt{\frac{8}{3}} \cos(x)\, e^{-x}$$

$$\psi'(x) = \sqrt{\frac{8}{3}} \left[(-\sin(x))\, e^{-x} - \cos(x)\, e^{-x} \right] =$$

$$= -\sqrt{\frac{8}{3}}\, e^{-x} \left(\cos(x) + \sin(x) \right) \checkmark$$

Επομένως η $\psi(x)$ είναι OK

$$\int_0^\infty \cos^2(x)\, e^{-2x}\, dx$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1$$

$$\cos^2(x) = \frac{\cos(2x) + 1}{2}$$

$$= \int_0^\infty \left(\frac{\cos(2x) + 1}{2}\right) e^{-2x}\, dx =$$

$$= \frac{1}{2}\int_0^\infty \cos(2x)\, e^{-2x}\, dx + \frac{1}{2}\int_0^\infty e^{-2x}\, dx$$

$$= \frac{1}{4}\int_0^\infty \cos(2x)\, e^{-2x}\, d(2x) + \frac{1}{2}\left(-\frac{1}{2}\right)\left(e^{-2x}\right)\Big|_0^\infty$$

$$\int_0^\infty \cos^2(y)\, e^{-2x}\, dx = \frac{1}{4}\int_0^\infty \cos(u)\, e^{-u}\, du + \frac{1}{4}$$

$$y = \sin(u) \Rightarrow dy = \cos(u)\, du$$

$$v = e^{-u} \Rightarrow dv = -e^{-u}\, du$$

$$vy\Big|_0^\infty = \int_0^\infty y\, du + \int_0^\infty u\, dy$$

$$\sin(u)\, e^{-u}\Big|_0^\infty = \int_0^\infty \sin(u)\left(-e^{-u}\right) du + \int_0^\infty e^{-u} \cos(u)\, du$$

$$\underbrace{\cancelto{0}{\frac{\sin(\infty)}{e^{\infty}}} - \cancelto{0}{\frac{\sin(0)}{e^{0}}}}_{0} = -\int_0^\infty \sin(u)\, e^{-u}\, du + \int_0^\infty \cos(u)\, e^{-u}\, du$$

$$=$$

$$\int_0^\infty \cos^2(x)\, e^{-2x}\, dx = \frac{1}{4} \int_0^\infty \cos(u)\, e^{-u}\, du + \frac{1}{4}$$

$$= \frac{1}{4} \underline{\int_0^\infty \sin(u)\, e^{-u}\, du} + \frac{1}{4}$$

$$y = \cos(u) \Rightarrow dy = -\sin(u)\, du$$

$$v = e^{-u} \Rightarrow dv = -e^{-u}\, du$$

$$yv\Big|_0^\infty = \int_0^\infty y\,dv + \int_0^\infty v\,dy$$

$$\cos(u)e^{-u}\Big|_0^\infty = \int_0^\infty \cos(u)(-e^{-u})\,du + \int_0^\infty e^{-u}(-\sin(u))\,du$$

$$\cancel{\frac{\cos(\infty)}{e^\infty}}^{\,0} - \frac{\cos(0)}{e^0} = -\int_0^\infty \cos(u)e^{-u}\,du - \int e^{-u}\sin(u)\,du$$

$=$

16d

$$-1 = -2\int e^{-u}\sin(u)\,du$$

$$\int_0^\infty \cos^2(x)\, e^{-2x}\, dx = \frac{1}{4}\int_0^\infty \cos(u)\, e^{-u}\, du + \frac{1}{4}$$

$$= \frac{1}{4}\underbrace{\int_0^\infty \cos(u)\, e^{-u}\, du}_{1/2} + \frac{1}{4}$$

$$= \frac{1}{4} \quad \frac{1}{2} + \frac{1}{4} = \frac{1}{8} + \frac{1}{4} = \frac{3}{8} \text{ !!}$$

3. Δίδεται η Κυματοσυνάρτηση που περιγράφει κάποια κατάσταση ενός σωματιδίου σε φρεάτιο:

$$\Psi(\mathrm{x}) = \sqrt{\frac{630}{a^9}}x^2(a-x)^2 \quad 0 \leq x \leq a$$

Βρείτε $\sigma_{\mathrm{E}}^2 = \langle E^2\rangle - \langle E\rangle^2$

$$\int_0^{a} \psi^*(x)\,\psi(x)\,dx = \frac{630}{a^9}\int_0^{a} x^2(a-x)^2 x^2(a-x)^2\,dx$$

$$= \frac{630}{a^9}\int_0^{a} x^4(a-x)^4\,dx =$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}, \quad (a+\beta)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}\beta^1 + \dots \binom{n}{n}\beta^n$$

$$(a-x)^4 = \binom{4}{0} a^4 + \binom{4}{1} a^3 (-x)^1 + \binom{4}{2} a^2 (-x)^2$$

$$+ \binom{4}{3} a^1 (-x)^3 + \binom{4}{4} a^0 (-x)^4$$

$$= \frac{4!}{0!\,4!} a^4 + \frac{4!}{1!\,3!} a^3 (-x) + \frac{4!}{2!\,2!} a^2 (-x)^2 +$$

$$\frac{4!}{3!\,1!} a (-x)^3 + \frac{4!}{4!\,0!} (-x)^4 =$$

$$= a^4 - 4a^3 x + 6a^2 x^2 - 4ax^3 + x^4$$

$$\frac{630}{a^9}\int_0^a x^4(a-x)^4\,dx =$$

$$\frac{630}{a^9}\int_0^a \left(a^4x^4 - 4a^3x^5 + 6a^2x^6 - 4ax^7 + x^8\right)dx$$

$$= \frac{630}{a^9}\left(\frac{a^4a^5}{5} - 4a^3\frac{a^6}{6} + 6\frac{a^2a^7}{7} - \frac{4a\,a^8}{8} + \frac{a^9}{9}\right)$$

$$= \frac{630}{\cancel{a^9}}\cancel{a^9}\left(\frac{1}{5} - \frac{2}{3} + \frac{6}{7} - \frac{1}{2} + \frac{1}{9}\right)$$

$$\int_0^{\alpha} \psi^{*}\psi \, dx = 630 \, \frac{378 - 1260 + 1620 - 945 + 216}{1896} = 1 \; \checkmark$$

Επομένως η $\psi(x)$ είναι κανονικοποιημένη.

$$\int_0^{\alpha} x^n (\alpha - x)^m \, dx = \frac{n! \, m! \, \alpha^{n+m+1}}{(m+n+1)!}$$

$$\frac{630}{\alpha^9} \int_0^{\alpha} x^4 (\alpha - x)^4 \, dx = \frac{630}{\alpha^9} \, \frac{4! \, 4!}{(4+4+1)!} \, \alpha^{4+4+1} = \frac{630 \times 4! \times 4!}{9!}$$

$$= \frac{362880}{9!} = \frac{9!}{9!} = 1 \; \checkmark$$

Για σωματίδιο σε φρέαρ $V(x)=0 \quad 0 \leq x \leq a$

και $$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

$$\langle E \rangle = \int_0^a \psi^* \hat{H} \psi \, dx \quad , \quad \hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

$$\langle E \rangle = \int_0^a \sqrt{\frac{630}{a^5}}\, x^2(a-x)^2 \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\left(\sqrt{\frac{630}{a^5}}\left(x^2(a-x)^2\right)\right) dx$$

$$\langle E \rangle = -\frac{630\hbar^2}{a^9\, 2m}\int_0^a x^2(a-x)^2 \frac{d}{dx}\left[2x(a-x)^2 + x^2\, 2(a-x)(-1)\right]dx$$

$$= -\frac{630\hbar^2}{2ma^9}\int_0^a x^2(a-x)^2\left[2(a-x)^2 + 2x\,2(a-x)(-1)\right.$$

$$\left. + 2x(-2)(a-x) + x^2(-2)(-1)\right]dx$$

$$= -\frac{630\hbar^2}{2ma^9}\int_0^a x^2(a-x)^2\left(2a^2 - 4ax + 2x^2 - 4ax + 4x^2 - 4ax + 4x^2 + 2x^2\right)dx$$

$$= -\frac{630 k^2}{2 m a^9} \int_0^a x^2 (a-x)^2 \left(2a^2 - 12 a x + 12 x^2\right) dx$$

$$= -\frac{630 k^2}{2 m a^9} \left[2a^2 \int_0^a x^2 (a-x)^2 dx - 12 a \int_0^a x^3 (a-x)^2 dx + 12 \int_0^a x^4 (a-x)^2 dx \right]$$

$$\int_0^a x^n (a-x)^m dx = \frac{n!\, m!\, a^{n+m+1}}{(m+n+1)!}$$

$$2a^2\int_0^a x^2(a-x)^2\,dx = 2a^2\,\frac{2!\,2!}{(2+2+1)!}\,a^{2+2+1} = \frac{8}{120}\,a^7 = \frac{a^7}{15}$$

$$12a\int_0^a x^3(a-x)^2\,dx = 12a\,\frac{3!\,2!}{(3+2+1)!}\,a^{3+2+1} = \frac{144}{720}\,a^7 = \frac{a^7}{5}$$

$$12\int_0^a x^4(a-x)^2\,dx = 12\,\frac{4!\,2!}{(4+2+1)!}\,a^{4+2+1} = \frac{576a^7}{5040} = \frac{4a^7}{35}$$

$$\langle E\rangle = -\frac{\hbar^2\,315\,a^7}{m\,a^3}\left(\frac{1}{15}-\frac{1}{5}+\frac{4}{35}\right) = -\frac{\hbar^2\,315}{ma^2}\,\frac{7-21+12}{105}$$

$$\langle E \rangle = \frac{\hbar^2}{m a^2} \frac{315 \times 2}{105} = 6 \frac{\hbar^2}{m a^2}$$

$$\langle E^2 \rangle = \int_0^a \psi^* \hat{H} \hat{H} \psi \, dx$$

$$= \int_0^a \sqrt{\frac{630}{a^9}} x^2 (a-x)^2 \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2}\right)\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2}\right)\left[\sqrt{\frac{630}{a^9}} x^2 (a-x)^2\right] dx$$

$$= \frac{\hbar^4}{4m^2} \frac{630}{a^9} \int_0^a x^2 (a-x)^2 \frac{d^2}{dx^2} \left(2a^2 - 12ax + 12x^2\right) dx$$

$$\langle E^2 \rangle = \frac{630}{4} \frac{\hbar^4}{m^2 a^9} \int_0^a x^2 (a-x)^2 \frac{d}{dx} \left(-12a + 24x\right) dx$$

$$= \frac{630 \hbar^4}{4 m^2 a^9} \int_0^a x^2 (a-x)^2 \, 24 \, dx$$

$$= 6 \, \frac{630}{m^2 a^9} \hbar^4 \, \frac{2! \, 2!}{(2+2+1)!} a^{2+2+1}$$

$$= \frac{6 \cdot 630 \cdot \hbar^4}{m^2 a^4} \frac{4}{120} = \frac{126 \hbar^4}{m^2 a^4}$$

$$\sigma_E^2 = \langle E^2 \rangle - \langle E \rangle^2$$

$$= \frac{126\, k^4}{m^2 a^4} - \left(\frac{6\, k^2}{m a^2} \right)^2$$

$$= \frac{k^4}{m^2 a^4} \left(126 - 36 \right) = \frac{90\, k^4}{m^2 a^4}$$

4. Ποιοι από τους ακόλουθους τελεστές είναι Ερμιτιανοί;

α) $\frac{d}{dx}$ $\beta)\, i\frac{d}{dx}$ $\gamma)\ \frac{d^2}{dx^2}$ $\delta)\ i\frac{d^2}{dx^2}$ $\varepsilon)\ x\frac{d}{dx}$ $\sigma\tau)\ x$

$$\int_{-\infty}^{\infty} \psi^* A \psi \, dx = \left(\int_{-\infty}^{\infty} \psi^* A \psi \, dx \right)^* = \int_{-\infty}^{\infty} \psi A^* \psi^* \, dx$$

$$\langle \psi | A \psi \rangle = \langle \psi A^* | \psi \rangle$$

α) $\int_{-\infty}^{\infty} \psi^* \frac{d}{dx} \psi \, dx = \int_{\psi(-\infty)}^{\psi(\infty)} \psi^* \, d\psi$

ολοκληρωση κατα παραγοντες $\psi^* \psi \Big|_{-\infty}^{\infty} = \int \psi^* d\psi + \int \psi \, d\psi^*$ $\Rightarrow$

$$\int \psi^* \, d\psi = \psi^* \psi \Big|_{-\infty}^{\infty} - \int \psi \, d\psi^*$$

$\psi^* \psi$ = πυκνότητα της πιθανότητας, (κανονική)

και επειδή $\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1 \Rightarrow \underline{\psi^* \psi(\infty) = \psi^* \psi(-\infty) = 0}$

$$\int \psi^* \, d\psi = -\int \psi \, d\psi^*$$ όχι Ερμιτιανός

(b) $\int \psi^* i \frac{d}{dx} \psi \, dx = \int \psi^* i \, d\psi = i \int \psi^* d\psi \quad (1)$

$\int \psi \left(-i \frac{d}{dx}\right) \psi^* dx = -i \int \psi \, d\psi^* \quad (2)$

$\underbrace{\psi \psi^* \Big|_{-\infty}^{\infty}}_{0} = \int \psi \, d\psi^* + \int \psi^* d\psi \Rightarrow$

$\int \psi \, d\psi^* = -\int \psi^* d\psi \quad (3)$

$\int \psi \left(-i \frac{d}{dx}\right) \psi^* dx \overset{(2)}{=} -i \left(-\int \psi^* d\psi\right) = i \int \psi^* d\psi = (1)$

άρα $i\, d/dx$ Ερμιτιανός

a) $$\int_{-\infty}^{\infty} \psi^* \frac{d^2}{dx^2} \psi \, dx = \int_{-\infty}^{\infty} \psi^* \frac{d}{dx} \underbrace{\left[\frac{d\psi}{dx} \right]}_{\phi(x)} dx$$

$$= \int_{-\infty}^{\infty} \psi^* \, d\phi(x)$$

used integrations

$$\psi^* \phi(x) \Big|_{-\infty}^{\infty} = \int_{-\infty}^{\infty} \psi^* \, d\phi(x) + \int_{-\infty}^{\infty} \phi(x) \, d\psi^* = 0$$

$$\psi^*(\infty) = \psi^*(-\infty) = 0$$

$$\int_{-\infty}^{\infty} \chi^* \frac{d^2}{dx^2}\psi \, dx = -\int_{-\infty}^{\infty} \phi \, d\psi^* = -\int_{-\infty}^{\infty} d\chi \, d\psi^* \quad 0$$

$$\int \psi \frac{d^2}{dx^2} \psi^* = \int \psi \frac{d}{dx} \underbrace{\left[\frac{d\psi^*}{dx} \right]}_{\phi^*(x)} dx = \int \psi \, d\phi^*$$

$\int$ κατά παράγοντες

$$\psi \phi^* \Big|_{-\infty}^{\infty} = \int \psi \, d\phi^* + \int \phi^* d\psi = 0$$

$$\psi(\infty) = \psi(-\infty) = 0 \qquad \int \psi \, d\phi^* = -\int \phi^* d\psi$$

$$\int \psi \frac{d^2}{dx^2} \psi^* = -\int d\psi^* \, d\psi = -\int d\psi^* \, d\psi$$

$$\overset{?}{=} \int \psi^* \frac{d^2}{dx^2} \psi$$

Ερμιτιανός

$$(\sigma) \quad \int \psi^* \, i \frac{d^2}{dx^2} \psi \, dx = i \int \psi^* \frac{d^2}{dx^2} \psi \, dx = \text{απτ} \; (\gamma)$$

$$= -i \int d\psi \, d\psi^*$$

$$\int \psi \left(-i \frac{d^2}{dx^2}\right) \psi^* dx = -i \int \psi \frac{d^2}{dx^2} \psi^* dx = -i \left(-\int d\psi^* d\psi\right)$$

$$= +i \int d\psi^* d\psi$$

ΟΧΙ Ερμιτιανός

E) $\int \psi^* x \frac{d}{dx} \psi \, dx = \int \underbrace{\psi^* x}_{\phi^*} d\psi = \int \phi^*(x) \, d\psi$

$$\phi^* \psi \Big]_{-\infty}^{\infty} = \int_{-\infty}^{\infty} \phi^* \, d\psi + \int_{-\infty}^{\infty} \psi \, d\phi^* = 0$$

$$\Rightarrow \int \phi^* d\psi = -\int_{-\infty}^{\infty} \psi \, d\phi^* = -\int_{-\infty}^{\infty} \psi \, d(\psi^* x)$$

$$\int \psi^* x \frac{d}{dx} \psi \, dx = -\int_{-\infty}^{\infty} \psi \, d(\psi^* x) = -\int_{-\infty}^{\infty} \psi \left(x \, d\psi^* + d\psi^* \right)$$

$$\int \psi^* x \frac{d}{dx}\psi \, dx = -\int \psi \, x \, d\psi^* - \int \psi \, d\psi^* \qquad 0$$

$$\int \psi \times \frac{d}{dx}\psi^* \, dx = \int \psi \times d\psi^* \overset{0}{=} -\int \psi^* \times \frac{d}{dx}\psi \, dx - \int \psi \, d\psi^*$$

Οχι Ερμιτιανός

6a)

$$\int \psi^{*} x \psi \, dx = \int \psi \, x \, \psi^{*} \, dx$$

$$x = x^{*}$$ Eppμιτιανός

5. Δείξτε ότι εάν ο τελεστής $\hat{A}$ είναι Ερμιτιανός τότε $\hat{A} - \langle \alpha \rangle$ είναι Ερμιτιανός αλλά και ότι το άθροισα δύο Ερμιτιανών τελεστών είναι επίσης Ερμιτιανός

$$\int \phi^* \hat{A} \psi \, dx = \int \psi \hat{A}^* \phi^* \, dx$$

$$\int \phi^* (\hat{A} - \langle \alpha \rangle) \psi \, dx = \int \phi^* \hat{A} \psi \, dx - \int \phi^* \langle \alpha \rangle \psi \, dx$$

$$= \int \psi \hat{A}^* \phi^* \, dx - \int \psi \langle \alpha \rangle \phi^* \, dx$$

$$= \int \psi (\hat{A}^* - \langle \alpha \rangle) \phi^* \, dx = \int \psi (\hat{A} - \langle \alpha \rangle)^* \phi^* \, dx$$

$$\langle \phi | (\hat{A} - \langle \alpha \rangle) \psi \rangle = \langle \phi (\hat{A} + \langle \alpha \rangle)^* | \psi \rangle \quad \checkmark$$

$\hat{A}$ και $\hat{B}$ Ερμιτιανοί

$$\int \phi^* (\hat{A} + \hat{B}) \psi \, dx = \int \phi^* \hat{A} \psi \, dx + \int \phi^* \hat{B} \psi \, dx$$

$$= \int \psi \hat{A}^* \phi^* \, dx + \int \psi \hat{B}^* \phi^* \, dx = \int \psi (\hat{A}^* + \hat{B}^*) \phi^* \, dx = \int \psi (\hat{A} + \hat{B})^* \phi^* \, dx$$

6. Βρείτε τον μεταθέτη $[\hat{A},\hat{B}]$ για τα ακόλουθα ζεύγη τελεστών

	$\hat{A}$	$\hat{B}$
α)	$\frac{d^2}{dx^2}$	x
β)	$\frac{d}{dx} - x$	$\frac{d}{dx} + x$
γ)	$\int_0^x dx$	$\frac{d}{dx}$
δ)	$\frac{d^2}{dx^2} - x$	$\frac{d}{dx} + x^2$

(α) $\left[\frac{d^2}{dx^2}, x\right]\psi =$

$= \left(\frac{d^2}{dx^2} x - x\frac{d^2}{dx^2}\right)\psi =$

$= \frac{d^2}{dx^2}(x\psi) - x\frac{d^2\psi}{dx^2} = \frac{d}{dx}\frac{d}{dx}(x\psi) - x\psi''$

$= \frac{d}{dx}(\psi + x\psi') - x\psi'' = \psi' + (x\psi')' - x\psi''$

$= \psi' + (\psi' + x\psi'') - x\psi'' = 2\psi' = 2\frac{d}{dx}\psi$

$\Rightarrow \left[\frac{d^2}{dx^2}, x\right] = 2\frac{d}{dx}$

6) $\left[\frac{d}{dy} - x, \frac{d}{dx} + x\right] =$

$$\left[\left(\frac{d}{dx} - y\right)\left(\frac{d}{dy} + x\right) - \left(\frac{d}{dy} + x\right)\left(\frac{d}{dx} - y\right)\right]\psi =$$

$$= \left(\frac{d^2}{dx^2} + \underbrace{\frac{d}{dy}x - x\frac{d}{dx}}_{\left[\frac{d}{dy}, x\right]} - x^2\right) - \left(\frac{d^2}{dx^2} \underbrace{- \frac{d}{dx}x + x\frac{d}{dx}}_{-\left[\frac{d}{dx}, x\right]} - x^2\right)$$

$$= \frac{d^2}{dx^2} + \left[\frac{d}{dx}, x\right] - x^2 - \frac{d^2}{dx^2} + \left[\frac{d}{dx}, x\right] + x^2$$

$$= 2\left[\frac{d}{dx}, x\right]$$

$$\left[\frac{d}{dx}-x,\ \frac{d}{dx}+x\right]\psi = 2\left[\frac{d}{dx},\ x\right]\psi$$

$$= 2\left(\frac{d}{dx}x - x\frac{d}{dx}\right)\psi = 2\left[\frac{d}{dx}(x\psi) - x\frac{d\psi}{dx}\right]$$

$$= 2\left[\left(\psi + x\frac{d\psi}{dx}\right) - x\frac{d\psi}{dx}\right] = 2\psi \implies$$

$$\left[\frac{d}{dx}-x,\ \frac{d}{dx}+x\right] = 2$$

a) $\left[\int_0^x dx, \frac{d}{dx}\right]\psi =$

$$\left(\int_0^x dx \frac{d}{dx} - \frac{d}{dx}\int_0^x dx\right)\psi =$$

$$= \int_0^x dx \frac{d\psi}{dx} - \frac{d}{dx}\int_0^x \psi dx =$$

$$= \int_{\psi(0)}^{\psi(x)} d\psi - \frac{d}{dx}\int_0^x \psi dx$$

Eg $\psi(x) = \frac{d f(x)}{dx} \Rightarrow \int \psi \, dx = \int \frac{df}{dx} dx = f(x) - f(0)$

$$\Rightarrow \frac{d}{dx}\left(\int \psi \, dx \right) = \frac{d f(x)}{dx} - \cancel{\frac{d f(0)}{dx}} \to 0$$

$$= \psi(x)$$

$$\left[\int_0^x dx, \frac{d}{dx} \right] \psi = \psi(x) - \psi(0) - \psi(x) = -\psi(0)$$

(d) $\left[\frac{d^2}{dx^2} - x , \frac{d}{dx} + x^2\right] \psi =$

$$\left(\frac{d^2}{dx^2} - x\right)\left(\frac{d}{dx} + x^2\right) - \left(\frac{d}{dx} + x^2\right)\left(\frac{d^2}{dx^2} - x\right)$$

$$= \cancel{\frac{d^3}{dx^3}} + \frac{d^2}{dx^2} x^2 - x\frac{d}{dx} \cancel{- x^3} - \cancel{\frac{d^3}{dx^3}} + \frac{d}{dx} x$$

$$- x^2 \frac{d^2}{dx^2} \cancel{+ x^3} = \left[\frac{d^2}{dx^2}, x^2\right] + \left[\frac{d}{dx}, x\right]$$

$$\left[\frac{d^2}{dx^2} - x , \frac{d}{dx} + x^2\right]\psi = \left[\frac{d^2}{dx^2} , x^2\right]\psi =$$

$$\left(\frac{d^2}{dx^2} x^2 - x^2 \frac{d^2}{dx^2}\right)\psi =$$

$$= \frac{d^2}{dx^2}(x^2\psi) - x^2 \frac{d^2\psi}{dx^2} = \frac{d}{dx}\left(2x\psi + x^2\psi'\right) - x^2\psi''$$

$$= 2\psi + 2x\psi' + 2x\psi' + x^2\psi'' - x^2\psi''$$

$$= 2\psi + 4x\psi''$$

$$\left(\frac{d}{dx}, x\right)\psi = \left(\frac{d}{dx}(x\psi) - x\frac{d}{dx}\psi\right)$$

$$= \psi + \cancel{x\frac{d\psi}{dx}} - \cancel{x\frac{d\psi}{dx}} = \psi$$

$$\left[\frac{d^2}{dx^2} - x, \frac{d}{dx} + x^2\right]\psi = 2\psi + 4x\psi'' + \psi$$

$$= 2\psi + 4\frac{d}{dx}\psi = \left(3 + 4x\frac{d}{dx}\right)\psi$$

$$\left[\frac{d^2}{dx^2} - x, \frac{d}{dx} + x^2\right] = \left(3 + 4x\frac{d}{dx}\right)$$

6. Βρείτε τις σταθερές των τριών πολυωνύμων ώστε αυτά να είναι ορθογώνια στο πεδίο τιμών $0 \le x \le 1$

$$f_0(x) = a_0 \qquad f_1(x) = a_1 + xb_1 \qquad f_2(x) = a_2 + xb_2 + x^2c_2$$

$$\int_0^1 f_0(x) f_1(x)\,dx = 0$$

$$\int_0^1 f_1(x) f_2(x)\,dx = 0$$

$$\int_0^1 f_1(x) f_2(x)\,dx = 0$$

$$\int_0^1 f_0(x) f_2(x) = 0$$

$$\int_0^1 f_0(x) f_0(x) = 1$$

$$\int_0^1 f_1(x) f_1(x) = 1$$

$$\int_0^1 f_2(x) f_2(x) = 1$$

6 εξισώσεις   6 αγνωστοι

$$\int_0^1 f_0^2(x)\,dx = \int_0^1 a_0^2\,dx = 1 \Rightarrow a_0^2 \int_0^1 dx = 1 \Rightarrow$$

$$a_0^2 = 1$$

Διαλέγοντας $a_0 = 1 \Rightarrow \boxed{f_0(x) = 1}$

$$\int_0^1 f_0 f_1\,dx = \int_0^1 (a_1 + b_1 x)\,dx = 0 \Rightarrow$$

$$a_1 \int_0^1 dx + b_1 \int_0^1 x\,dx = 0 \Rightarrow a_1 + \frac{b_1}{2} = 0 \Rightarrow$$

$$a_1 = -\frac{b_1}{2} \qquad f_1(x) = a_1 - 2a_1 x = a_1(1-2x)$$

$$\int_0^1 f_1^2\,dx = 1 \Rightarrow \int_0^1 a_1^2 (1-2x)^2\,dx = 1$$

$$a_1^2 \int_0^1 (1 - 4x + 4x^2)\,dx = 1$$

$$a_1^2 \left(1 - \frac{4}{2} + \frac{4}{3}\right) = 1 \Rightarrow a_1^2 \left(-1 + \frac{4}{3}\right) = 1$$

$$a_1^2 \frac{1}{3} = 1 \Rightarrow a_1 = \pm\sqrt{3}$$

Διαλέγουμε $\boxed{f_1 = \sqrt{3}\,(1-2x)}$

$$\int_0^1 f_0 f_2 \, dx = 0 \Rightarrow \int_0^1 1\,(a_2 + b_2 x + c_2 x^2)\,dx = 0$$

$$a_2 \int_0^1 dx + b_2 \int_0^1 x\,dx + c_2 \int_0^1 x^2\,dx = 0$$

$$a_2 + \frac{b_2}{2} + \frac{c_2}{3} = 0 \qquad (1)$$

$$\int_0^1 f_1 f_2 \, dx = 0 \Rightarrow \int_0^1 \sqrt{3}\,(1 - 2x)\,(a_2 + b_2 x + c_2 x^2)\,dx = 0$$

$$\underbrace{\int_0^1 (a_2 + b_2 x + c_2 x^2)\,dx}_{0} - 2\int (a_2 x + b_2 x^2 + c_2 x^3)\,dx = 0$$

$$a_2\int_0^1 x\,dx + b_2\int_0^1 x^2\,dx + c_2\int_0^1 x^3\,dx = 0$$

$$\frac{a_2}{2} + \frac{b_2}{3} + \frac{c_2}{4} = 0 \qquad (2)$$

$$\frac{a_2}{2} + \frac{b_2}{\cancel{4}\,2} + \frac{c_2}{6\,\cancel{3}} = 0 \qquad (1)$$

$$(2)-(1) \qquad b_2\left(\frac{1}{3}-\frac{1}{4}\right) + c_2\left(\frac{1}{4}-\frac{1}{6}\right) = 0$$

$$b_2\,\frac{1}{12} + c_2\,\frac{1}{12} = 0 \Rightarrow \boxed{b_2 = -c_2}$$

(1) $$a_2 + \frac{b_2}{2} - \frac{b_2}{3} = 0$$

$$a_2 + b_2\left(\frac{3-2}{6}\right) = 0 \Rightarrow$$

$$a_2 = -\frac{b_2}{6}$$

$$f_2(x) = a_2 - 6a_2 x + 6a_2 x^2 = a_2\left(1 - 6x + 6x^2\right)$$

$$\int_0^1 f_2^2(x)\,dx = 1 = a_2^2 \underbrace{\int_0^1 \left(1 - 6x + 6x^2\right)^2 dx}_{1/5} = 1$$

https://www.wolframalpha.com/input/?i=integrate%280%2C1%29+%281-6x%2B6x%5E2%29%5E2

$$\frac{a_2^2}{5} - 1 = 0 \Rightarrow a_2 = \pm\sqrt{5}$$

Ainsi, j'ajoute un $\sqrt{5}$ $\Rightarrow$

$$\boxed{f_2 = \sqrt{5}\left(1 - 6x + 6x^2\right)}$$

7. Δίδονται οι ιδιοτιμές και ιδιοσυναρτήσεις ενός τελεστή $\hat{A}$: $\hat{A}|n\rangle = a_n|n\rangle$

Δείξτε ότι μια κυματοσυνάρτηση $|\phi\rangle$ μπορεί να εκφραστεί ως γραμμικός συνδυασμός των $|n\rangle$, $|\phi\rangle = \sum_n c_n|n\rangle$ όπου $c_n = \langle n|\phi\rangle$. Αντίστοιχα $\langle\phi| = \sum_n c_n^*\langle n|$ και $c_n^* = \langle\phi|n\rangle$. Ο όρος $c_n^* c_n$ είναι η πιθανότητα μέτρησης της ιδιοτιμής a_n όταν ο $\hat{A}$ δράσει πάνω στην $|\phi\rangle$.

Δείξτε ότι $|\phi\rangle = \sum_n |n\rangle\langle n|\phi\rangle$ και $\langle\phi| = \sum_n \langle\phi|n\rangle\langle n|$

Και αυτό συνεπάγεται

$$\sum_n |n\rangle\langle n| = 1$$

Σύμφωνα με τον Fourier, εφόσον τα $|n\rangle$ αποτελούν "πλήρες" σετ των ιδιοσυναρτήσεων του $\hat{A}$, τότε οποιαδήποτε κυματοσυνάρτηση $|\phi\rangle$ δύναται να γραφτεί ως γραμμικός συνδυασμός των $|n\rangle$: $\phi = \sum_n c_n \psi_n$

$$\times \psi_n^* : \quad \psi_m^* \phi = \psi_m^* \sum_n c_n \psi_n = \sum_n c_n \psi_m^* \psi_n$$

$$\int \quad dx : \quad \int \psi_m^* \phi \, dx = \int \left(\sum_n c_n \psi_m^* \psi_n \right) dx$$

$$= \sum_n c_n \underbrace{\int \psi_m^* \psi_n \, dx}_{\delta_{mn}}$$

$$\delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

$$= \sum_n c_n \delta_{mn} = c_m \Rightarrow$$

$$c_m = \int \psi_m^* \phi \, dx = \langle m | \phi \rangle$$

$$c_m^* = (\langle m | \phi \rangle)^* = \langle \phi | m \rangle$$

$$\left.\begin{aligned}\phi &\equiv |\phi\rangle \\ \psi_n &\equiv |n\rangle\end{aligned}\right\} \Rightarrow |\phi\rangle = \sum_n c_n |n\rangle$$

$$= \sum_n \langle n|\phi\rangle |n\rangle$$

αλλά $\langle n|\phi\rangle$ είναι απλά ένας αριθμός

επομένως $\langle n|\phi\rangle |n\rangle = |n\rangle\langle n|\phi\rangle$

$$|\phi\rangle = \sum_n |n\rangle\langle n|\phi\rangle = \left(\sum_n |n\rangle\langle n|\right)|\phi\rangle$$

επομένως $\boxed{\sum_n |n\rangle\langle n| = 1}$

8. Για ένα σωματίδιο σε φρεάτιο βρείτε την μέση θέση $\langle x \rangle$ όταν η κατάστασή του περιγράφεται από την κυματοσυνάρτηση

$$\Psi(x,t) = \left(\frac{1}{a}\right)^{1/2} e^{-iE_2 t/\hbar} \sin\frac{2\pi x}{a} + \left(\frac{1}{a}\right)^{1/2} e^{-iE_3 t/\hbar} \sin\frac{3\pi x}{a}$$

Όπου a είναι το εύρος του πηγαδιού. Βρείτε το πλάτος της ταλάντωσης και την συχνότητα πέριξ το φυσικού μέσου δηλ. a/2. Πόση είναι η μέση ενέργεια του σωματιδίου;

Οι ιδιοσυναρτήσεις για το σωματίδιο σε φρεάτιο είναι $|n\rangle = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$

$$\Psi(x,t) = |n\rangle e^{-\frac{iE_n}{\hbar}t}$$

$$E_n = n^2 E_1 \ , \ E_1 = \frac{h^2}{8ma^2}$$

$$\psi(x,t) = \frac{1}{\sqrt{2}}\psi_2(x,t) + \frac{1}{\sqrt{2}}\psi_3(x,t)$$

$$\langle\psi|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle\psi_2| + \frac{1}{\sqrt{2}}\langle\psi_3|\right)$$

$$\left(\frac{1}{\sqrt{2}}|\psi_2\rangle + \frac{1}{\sqrt{2}}|\psi_3\rangle\right)$$

$$= \frac{1}{2}\left(\cancelto{0}{\langle\psi_2|\psi_3\rangle} + \overbrace{\langle\psi_2|\psi_2\rangle}^{1} + \cancelto{0}{\langle\psi_3|\psi_2\rangle}\right.$$

$$\left.\underbrace{\langle\psi_3|\psi_3\rangle}_{1}\right)$$

$= 1$ knovikottoint ivm

$$\langle x \rangle = \langle \psi | x | \psi \rangle =$$

$$= \left(\frac{1}{\sqrt{2}} \langle \psi_2 | + \frac{1}{\sqrt{2}} \langle \psi_3 | \right) x \left(\frac{1}{\sqrt{2}} | \psi_2 \rangle + \frac{1}{\sqrt{2}} | \psi_3 \rangle \right)$$

$$= \frac{1}{2} \Big(\overbrace{\langle \psi_2 | x | \psi_2 \rangle}^{\alpha/2} + \langle \psi_2 | x | \psi_3 \rangle +$$

$$\langle \psi_3 | x | \psi_2 \rangle + \underbrace{\langle \psi_3 | x | \psi_3 \rangle}_{\alpha/2} \Big)$$

$$= \frac{1}{2} \Big(\alpha + \langle \psi_3 | x | \psi_2 \rangle + \langle \psi_2 | x | \psi_3 \rangle \Big)$$

$$\langle \psi_3 | x | \psi_2 \rangle + \langle \psi_2 | x | \psi_3 \rangle =$$

$$= e^{i\frac{E_3}{\hbar}t} \langle 3 | x | 2 \rangle e^{-i\frac{E_2}{\hbar}t} + e^{i\frac{E_2}{\hbar}t} \langle 2 | x | 3 \rangle e^{-i\frac{E_3}{\hbar}t}$$

αλλά $\langle 3 | x | 2 \rangle \underset{\substack{\uparrow \\ \text{Ερμιτιανότ}}}{=} \left[\langle 3 | x | 2 \rangle\right]^* = \langle 2 | x^* | 3 \rangle$

$$= \langle 2 | x | 3 \rangle$$

$$x = x^*$$

Επομένως

$$\langle \psi_3 | x | \psi_2 \rangle + \langle \psi_2 | x | \psi_3 \rangle = \langle 3 | x | 2 \rangle \left(e^{i\frac{E_3}{\hbar}t} e^{-i\frac{E_2}{\hbar}t} + e^{i\frac{E_2}{\hbar}t} e^{-i\frac{E_3}{\hbar}t} \right)$$

$$= \langle 3|x|2\rangle \left(e^{\frac{i(E_3-E_2)}{\hbar}t} + e^{-\frac{i(E_3-E_2)}{\hbar}t} \right)$$

$$\omega = \frac{E_3-E_2}{\hbar} \Rightarrow$$

$$= \langle 3|x|2\rangle \underbrace{\left(e^{i\omega t} + e^{-\omega t} \right)}_{2\cos(\omega t)}$$

$$\langle 3|x|2\rangle = \int_0^a \left[\sqrt{\frac{2}{a}}\sin\left(\frac{3\pi x}{a}\right)\right] x \left[\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right)\right] dx$$

$$= \frac{2}{a}\int_0^a x\sin\left(\frac{3\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) dx =$$

$$= \frac{2}{a}\left(-\frac{24a^2}{25\pi^2}\right) = -\frac{48}{25}\frac{a}{\pi^2}$$

https://www.wolframalpha.com/input/?i=integrate%280%2C+a%29+%28x*sin%282*PI*x%2Fa%29*sin%283*Pi*x%2Fa%29%29

$$\langle \psi_3 | x | \psi_2 \rangle + \langle \psi_2 | x | \psi_3 \rangle =$$

$$-\frac{48a}{25\pi^2}\left(2\cos(\omega t)\right) \Rightarrow$$

$$\boxed{\langle x \rangle = \frac{a}{2} - \frac{48a}{25\pi^2}\cos(\omega t)}$$

συχνότητα $\omega = \frac{E_3 - E_2}{\hbar}$ $\omega = 2\pi\nu$

$$\boxed{\nu = \frac{E_3 - E_2}{2\pi \; h/2\pi} = \frac{E_3 - E_2}{h}}$$

$t = 0 \Rightarrow \langle x(0) \rangle = \frac{\alpha}{2} - \frac{48}{25\pi^2} \alpha$

$t = \frac{T}{2} \Rightarrow \omega \frac{T}{2} = \pi$

$$\langle x(\pi) \rangle = \frac{\alpha}{2} - \frac{48}{25\pi^2} \alpha \underbrace{\cos(\pi)}_{-1}$$

$$= \frac{\alpha}{2} + \frac{48 \alpha}{25\pi^2}$$

$$A = \langle x(\pi) \rangle - \langle x(0) \rangle = 2 \frac{48 \alpha}{25\pi^2}$$

$$= \frac{96 \alpha}{25\pi^2}$$

$$\langle E \rangle = \langle \psi | \hat{H} | \psi \rangle$$

$$= \left[\frac{1}{\sqrt{2}} \left(\langle \psi_3 | + \langle \psi_2 | \right) \right] \hat{H} \left[\frac{1}{\sqrt{2}} \left(| \psi_2 \rangle + | \psi_3 \rangle \right) \right]$$

$$= \frac{1}{2} \langle \psi_3 | \hat{H} | \psi_3 \rangle + \frac{1}{2} \langle \psi_2 | \hat{H} | \psi_2 \rangle$$

$$+ \frac{1}{2} \langle \psi_3 | \hat{H} | \psi_2 \rangle + \frac{1}{2} \langle \psi_2 | \hat{H} | \psi_3 \rangle$$

$$\boxed{\hat{H} | \psi_n \rangle = E_n | \psi_n \rangle}$$

$$\langle E \rangle = \frac{1}{2} E_3 \overbrace{\langle \psi_3 | \psi_3 \rangle}^{1} + \frac{1}{2} E_2 \overbrace{\langle \psi_2 | \psi_2 \rangle}^{1}$$

$$+ \frac{1}{2} E_2 \cancelto{0}{\langle \psi_3 | \psi_2 \rangle} + \frac{1}{2} E_3 \cancelto{0}{\langle \psi_2 | \psi_3 \rangle}$$

$$\boxed{\langle E \rangle = \frac{1}{2} E_3 + \frac{1}{2} E_2}$$

9. Η κυματοσυνάρτηση $\Psi_Q(x,t)$ είναι γραμμικός συνδυασμός των ιδιοσυναρτήσεων του σωματιδίου σε φρεάτιο $\Psi_n(x,t)$

$$\Psi_Q(x,t) = \sqrt{\frac{i}{4}}\Psi_1(x,t) - \sqrt{\frac{-1}{6}}\Psi_3(x,t) + \frac{1}{2}\Psi_4(x,t) + c_5\Psi_5(x,t),$$

Βρείτε τον συντελεστή c_5 και την μέση ενέργεια $\langle \mathrm{E}_Q \rangle$.

Για να βρούμε το C_5 χρησιμοποιούμε την συνθήκη κανονικοποίησης

$$\langle Q | Q \rangle = 1$$

$$\langle Q | Q \rangle = \sum_{n=1}^{5} \langle Q | n \rangle \langle n | Q \rangle$$

$$= \langle Q | 1 \rangle \langle 1 | Q \rangle + \langle Q | 3 \rangle \langle 3 | Q \rangle +$$

$$+ \langle Q | 4 \rangle \langle 4 | Q \rangle + \langle Q | 5 \rangle \langle 5 | Q \rangle$$

Προσοχή $\sqrt{\frac{-1}{6}} = \frac{\sqrt{-1}}{\sqrt{6}} = \frac{i}{6}$

$\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \Rightarrow$

Βρείτε ασκήσεις για γραφικού

$\frac{\sqrt{i}}{2} = \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} i \Rightarrow \left(\frac{\sqrt{i}}{2}\right)^2 = \frac{1}{4 \cdot 2} + \frac{1}{4 \cdot 2} = \frac{2}{8} = \frac{1}{4}$

$-\frac{i}{\sqrt{6}} \frac{i}{\sqrt{6}} = \frac{-i^2}{(\sqrt{6})^2} = \frac{-(-1)}{6} = \frac{1}{6}$

$1 = \frac{1}{4} + \frac{1}{6} + \frac{1}{4} \quad |\,|c_5|^2 \Rightarrow |c_5|^2 = 1 - \frac{2}{4} - \frac{1}{6}$

$|c_5|^2 = \frac{12-6-2}{12} = \frac{4}{12} = \frac{1}{3} \qquad c_5 = \pm\frac{1}{\sqrt{3}}$

$$\langle E_Q \rangle = \sum_i \langle Q|n\rangle\langle n|Q\rangle E_n$$

$$\boxed{\langle E_Q \rangle = \sum_n C_n^2 E_n}$$

$$\langle E_Q \rangle = \frac{1}{4} E_1 + \frac{1}{6} E_3 + \frac{1}{4} E_4 + \frac{1}{3} E_5$$

$$= \frac{1}{4} E_1 + \frac{1}{6} 3^2 E_1 + \frac{1}{4} 4^2 E_1 + \frac{1}{3} 5^2 E_1$$

$$= E_1 \left(\frac{1}{4} + \frac{9}{6} + \frac{16}{4} + \frac{25}{3} \right)$$

$$\langle E_Q \rangle = \frac{3 + 18 + 48 + 100}{12} E_1 = \frac{163}{12} E_1$$

10. Βρείτε τους συντελεστές της κυματοσυνάρτησης $\Psi_C(x,t)$ είναι γραμμικός συνδυασμός των ιδιοσυναρτήσεων του σωματιδίου σε φρεάτιο $\Psi_n(x,t)$ ώστε $\langle E_C \rangle = 4E_1$

$$\Psi_C = c_1\Psi_1 + c_2\Psi_2 + c_3\Psi_3$$

Από τη συνθήκη κανονικοποίησης

$$c_1^2 + c_2^2 + c_3^2 = 1 \qquad (1)$$

και

$$\langle E_C \rangle = c_1^2 E_1 + c_2^2 E_2 + c_3^2 E_3$$

$$E_n = n^2 E_1$$

$$\langle E_C \rangle = 4E_1 = c_1^2 E_1 + c_2^2 2^2 E_1 + c_3^2 3^2 E_1$$

$$4 = c_1^2 + 4c_2^2 + 9c_3^2 \qquad (2)$$

$(1) \times 4 \qquad 4c_1^2 + 4c_2^2 + 4c_3^2 = 4$

$(2) \qquad c_1^2 + 4c_2^2 + 9c_3^2 = 4$

$4 \times (1) - (2)$

$$3c_1^2 - 5c_3^2 = 0 \Rightarrow$$

$$3c_1^2 = 5c_3^2 \Rightarrow$$

$$c_1 = \pm\sqrt{\frac{5}{3}}\, c_3$$

Διαλέγουμε $c_1 = +\sqrt{\frac{5}{3}}\, c_3 \Rightarrow$ αντικαθιστώντας στην (1)

$$\frac{5}{3} c_3^2 + c_2^2 + c_3^2 = 1 \Rightarrow$$

$$\frac{8}{3} c_3^2 = 1 - c_2^2$$

Equidistant $c_1 = \sqrt{\frac{3}{5}} \Rightarrow$

$$c_3 = \sqrt{\frac{3}{5}} \sqrt{\frac{3}{5}} = \frac{3}{5}$$

$$\frac{3}{5} + c_2^2 + \frac{9}{25} = 1 \Rightarrow$$

$$c_2^2 = 1 - \frac{9}{25} - \frac{3}{5} = \frac{25 - 9 - 15}{25} = \frac{1}{25}$$

$$c_2 = \frac{1}{\sqrt{25}} = \frac{1}{5}$$

$$\psi_c = \sqrt{\frac{3}{5}}\,\psi_1 + \frac{1}{5}\,\psi_2 + \frac{3}{5}\,\psi_3$$

Επιβεβαίωση:

$$c_1^2 + c_2^2 + c_3^2 = 1 \Rightarrow \frac{3}{5} + \frac{1}{25} + \frac{9}{25} = \frac{15+1+9}{25} = \frac{25}{25} = 1 \checkmark$$

$$\langle E \rangle = 4E_1 \Rightarrow \quad 4 = c_1^2 + 4c_2^2 + 9c_3^2$$

$$\frac{3}{5} + 4\,\frac{1}{25} + 9\,\frac{9}{25} = \frac{15+4+81}{25} =$$

$$= \frac{100}{25} = 4 \checkmark\checkmark$$

11. Εάν η $\Psi_\mathrm{D}(x,t)$ που είναι γραμμικός συνδυασμός των ιδιοσυναρτήσεων του σωματιδίου σε φρεάτιο $\Psi_n(x,t)$ (α) βρείτε τις 3 πιθανές τιμές του συντελεστή c_3. (β) Διαλέξτε μία από τις τιμές του c_3 και βρείτε την πιθανότητα για τις ενέργειες E_1 E_2 E_3 (γ) βρείτε την $\delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2}$ σαν συνάρτηση του E_1

(α)

$$\Psi_D(x,t) = \sqrt{\frac{i}{4}}\Psi_1(x,t) - c_3\Psi_3(x,t).$$

Από την συνθήκη κανονικοποίησης

$$c_1^2 + c_3^2 = 1$$

$$c_1 = \sqrt{\frac{i}{4}} = \frac{\sqrt{i}}{2} \quad \Bigg\} \quad c_1^* c_1 = \frac{\sqrt{-i}}{2}\,\frac{\sqrt{i}}{2} = \frac{(-i^2)^{1/2}}{4} = \frac{1}{4}$$

$$c_1^* = \frac{\sqrt{-i}}{2}$$

$$\sqrt{i} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \Rightarrow \frac{\sqrt{i}}{2} = \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\,i \Rightarrow \left(\frac{\sqrt{i}}{2}\right)^2 = \left(\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2$$

$$= \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$$

$$\frac{1}{4} + c_3^2 = 1 \Rightarrow c_3^2 = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow$$

$$c_3 = \pm \frac{\sqrt{3}}{2} \quad \text{și} \quad c_3 = -i\frac{\sqrt{3}}{2}$$

b) $c_3 = \frac{\sqrt{3}}{2} \Rightarrow$

$$\boxed{\psi_0 = \frac{\sqrt{1}}{2}\psi_1 - \frac{\sqrt{3}}{2}\psi_3}$$

$$\langle E \rangle = \sum_n c_n^2 E_n = \sum_n P_n E_n \Rightarrow$$

$$\boxed{P_n = c_n^2}$$

$$P(E_1) = \left(\frac{\sqrt{1}}{2}\right)^2 = 1/4$$

$$P(E_2) = 0$$

$$P(E_3) = 3/4$$

(a) $\langle E \rangle = P_1 E_1 + P_3 E_3$

$= \frac{1}{4} E_1 + \frac{3}{4} E_3$ với $E_n = n^2 E_1$

$= \frac{1}{4} E_1 + \frac{3}{4} 3^2 E_1 \Rightarrow$

$\langle E \rangle = E_1 \left(\frac{1}{4} + \frac{27}{4} \right) = \frac{28}{4} E_1 = 7 E_1$

$\langle E^2 \rangle = \langle \psi_0 | \hat{H}^2 | \psi_0 \rangle =$

$= (c_1 \langle \psi_1 | + c_3 \langle \psi_3 |) \hat{H}^2 (c_1 | \psi_1 \rangle + c_3 | \psi_3 \rangle)$

$= c_1^2 \langle \psi_1 | \psi_1 \rangle E_1^2 + c_3 c_1 \cancel{\langle \psi_3 | \psi_1 \rangle}^{0} E_1^2$

$+ c_3^2 \langle \psi_3 | \psi_3 \rangle E_3^2 + c_1 c_3 \cancel{\langle \psi_1 | \psi_3 \rangle}^{0} E_3^2$

$$\langle E^2 \rangle = c_1^2 E_1^2 + c_3^2 E_3^2$$

$$= \frac{1}{4} E_1^2 + \frac{3}{4} E_3^2$$

$$= \frac{1}{4} E_1^2 + \frac{3}{4} \left(3^2 E_1\right)^2$$

$$= E_1^2 \left(\frac{1}{4} + \frac{3}{4} 81 \right)$$

$$= 61 E_1^2$$

$$\delta E = \sqrt{61 E_1^2 - (7 E_1)^2} = E_1 \sqrt{12} = 2\sqrt{3} E_1$$