A–1. Find the real and imaginary parts of the following quantities:

(a) $(2-i)^3$ (b) $e^{\pi i/2}$

(c) $e^{-2+i\pi/2}$ (d) $(\sqrt{2}+2i)e^{-i\pi/2}$

$$(a)\ (2-i)^3 = 2^3 - 3.2^2.i + 3.2.i^2 - i^3 = 8 - 12i + 6(-1) - i^2.i$$
$$= 8-6 -12i + i = 2-11i$$

$$(b)\ e^{\pi i/2} = \cos\frac{\pi}{2} + i\sin\pi/2 = 0 + i = i$$

$$c)\ e^{-2+i\pi/2} = e^{-2} \cdot e^{i\pi/2} = \frac{i}{e^2}$$

$$d)\ (\sqrt{2}+2i)(\cos(-\pi/2) + i\sin(-\pi/2)) = (\sqrt{2}+2i)(i.(-1))$$
$$= -i\sqrt{2} - 2i^2 = 2 - i\sqrt{2}$$

A–4. Express the following complex numbers in the form $x + iy$:

(a) $e^{\pi/4i}$ (b) $6e^{2\pi i/3}$

(c) $e^{-(\pi/4)i+\ln 2}$ (d) $e^{-2\pi i} + e^{4\pi i}$

a) $e^{\frac{\pi}{4i}} = e^{\frac{\pi \cdot i}{4 i^2}} = e^{-\frac{\pi}{4} i}$

$= \cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) =$

$= \frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$

(b) $6 e^{\frac{2\pi}{3} i} = 6\left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right) = 6\left(-\frac{1}{2} + i \frac{\sqrt{3}}{2}\right)$

$= -3 + 3\sqrt{3}\, i$

(c) $e^{\ln 2}\left(e^{-\frac{\pi}{4} i}\right) = 2\left(\cos\left(-\pi/4\right) + i \sin\left(-\pi/4\right)\right)$

$= 2\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right) = \sqrt{2} - \sqrt{2}\, i$

d) $[\cos(-2\pi) + i \sin(-2\pi)] + (\cos(4\pi) + i \sin(4\pi))$

$= 1 + 0 + 1 + 0 = 2$

A–13. Evaluate i^i.

$i = \quad r = 1$

$\sin(\theta) = \frac{1}{1} = 1 \Rightarrow \theta = \pi/2 \Big\} \; i = e^{i\pi/2}$

$$i^i = \left(e^{i\pi/2}\right)^i = e^{i^2 \pi/2} = e^{-\pi/2}$$

1. (10 points) - (a) Find z, defined as the square root of $-i$. (b) Write that z as $z = Rez + i\ Imz$ (real and imaginary parts of z), and then find the real and imaginary parts for the square root of z^*. Don't forget that every number (real or complex) has two square roots.

(a) $z = \sqrt{-i} = \sqrt{-1}\sqrt{i} = i\, i^{1/2} = i^{3/2}$

(b) To do this we want to find $z = a + bi$

$a = Re(z)$, $b = Im(z)$

since $z = i\, i^{1/2}$ we need to evaluate $i^{1/2}$. You can find this online or let https://www.youtube.com/watch?v=W7Pl2opRVzE

$i^{1/2} = x + iy \Rightarrow \underbrace{i} = (x+iy)^2 = x^2 - y^2 + \underbrace{2xyi}$

So since the left hand side is i only the "terms with i" on the right side survive $\Rightarrow$ $x^2 - y^2 = 0$ and $i = 2xyi$

$x = \pm y$ $\quad 2xy = 1 \Rightarrow$ $\boxed{\pm y^2 = \frac{1}{2}}$ This only works for $\underline{+}$ $y^2 = \frac{1}{2} \Rightarrow y = \pm\frac{1}{\sqrt{2}}$

$= x$

So $\sqrt{i} = \pm\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right) \Rightarrow$

$$z = i\sqrt{i} = \pm\left(i\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i^2\right) \Rightarrow$$

$$z = \pm\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right) \quad \text{or}$$

$$z = \pm\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)$$

CASE A

$$z = +\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) = \overset{x}{-\frac{\sqrt{2}}{2}} + \overset{y}{\frac{\sqrt{2}}{2}}\, i$$

$$r = \sqrt{z^* z} = \sqrt{\underbrace{\left(\frac{\sqrt{2}}{2}\right)^2}_{x^2} + \underbrace{\left(\frac{\sqrt{2}}{2}\right)^2}_{y^2}} = \sqrt{\frac{2}{4} + \frac{2}{4}} = 1$$

$$x = r \cdot \cos\theta \Rightarrow -\frac{\sqrt{2}}{2} = \cos\theta \Rightarrow \boxed{\theta = \frac{3\pi}{4}}$$

$$z = e^{i\frac{3\pi}{4}}$$

$$z^* = e^{-i\frac{3\pi}{4}}$$

$$\sqrt{z^*} = e^{-i\frac{3\pi}{8}}$$

$$\mathrm{Re}(\sqrt{z^*}) = \cos\left(-\frac{3\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2} \quad (\text{symbolab})$$

$$\mathrm{Im}(\sqrt{z^*}) = \sin\left(-\frac{3\pi}{8}\right) = -\frac{\sqrt{2+\sqrt{2}}}{2}$$

CASE B

$$z = -\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$$

$$r = 1, \quad r\cos\theta = \frac{\sqrt{2}}{2} \Rightarrow \cos\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}$$

$$z = e^{i\frac{\pi}{4}}, \quad z^* = e^{-i\frac{\pi}{4}} \Rightarrow \sqrt{z^*} = e^{-i\pi/8}$$

$$\mathrm{Re}(\sqrt{z^*}) = \cos\left(-\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\mathrm{Im}(\sqrt{z^*}) = \sin\left(-\pi/8\right) = -\frac{\sqrt{2-\sqrt{2}}}{2}$$

2. (5 points) - Use the Euler Formula derived in Lecture to evaluate the real and imaginary parts of the complex wave function $\psi(x) = 2e^{ikx}$ for these 5 values of x: $x = \lambda/2,\ \lambda/3,\ \lambda/4,\ 3\lambda + \lambda/5,\ 13\lambda/6$. You'll have to recall the standard relation between wave vector k and wavelength λ, and evaluate some trig functions (no more than 2 sign. figs.).

$$k = \frac{2\pi}{\lambda}$$

$$y(x) = 2e^{i\frac{2\pi}{\lambda}x}$$

$$x = \lambda/2 \Rightarrow \quad y(\lambda/2) = 2e^{i\frac{2\pi}{\lambda}\cdot\frac{\lambda}{2}} = 2e^{i\pi}$$

$$\mathrm{Re}(y) = 2\cos(\pi) = -2$$

$$\mathrm{Im}(y) = 2\sin(\pi) = 0$$

$$x = \lambda/3 \Rightarrow \quad y(\lambda/3) = 2e^{i\frac{2\pi}{\lambda}\frac{\lambda}{3}} = 2e^{i\frac{2\pi}{3}}$$

$$\mathrm{Re}(y) = 2\cos\left(\frac{2\pi}{3}\right) = -1$$

$$\mathrm{Im}(y) = 2\sin\left(\frac{2\pi}{3}\right) = 1.7$$

$x = \lambda/4 \qquad y(\lambda/4) = 2e^{i\frac{2\pi}{\lambda}\frac{\lambda}{4}} = 2\,e^{i\,\pi/2}$

$\mathrm{Re}\, y = 2\cos(\pi/2) = 0$

$\mathrm{Im}(y) = 2\sin(\pi/2) = 2$

$x = 3\lambda + \lambda/5\,, \quad y = 2\,e^{i\frac{2\pi}{\lambda}\left(3\lambda + \frac{\lambda}{5}\right)}$

$= 2\,e^{i\,2\pi\left(\frac{16}{5}\right)} = 2^{i\,\frac{32}{5}\pi}$

$\mathrm{Re}(y) = 2\cos\left(\frac{32\pi}{5}\right) = 0{,}62$

$\mathrm{Im}(y) = 2\sin\left(\frac{32\pi}{5}\right) = 1{,}9$

$x = 13\frac{\lambda}{6}\,, \quad y = 2e^{i\frac{2\pi}{\lambda}\,13\frac{\lambda}{6}} \Rightarrow y = 2\,e^{i\frac{13}{3}\pi}$

$\mathrm{Re}\, y = 2\cos\left(\frac{13\pi}{3}\right) = 1 \qquad \mathrm{Im}(y) = 2\sin\left(\frac{13\pi}{3}\right) = 1{,}7$

3. (10 points) - Is it true that $(\sqrt{z}\,)^* = \sqrt{z^*}$? Use Euler's formula to help prove your answer, yes or no. Start by using the fact that a complex z can always be written in terms of its real and imaginary parts as $a + ib$, with real numbers a and b. Then express z in terms of its absolute value $|z| \equiv r = +\sqrt{a^2 + b^2}$ (which is its positive "length" as a vector in the complex plane), and its angle θ in the plane, and so $z = |z|e^{i\theta}$). Go from there and don't forget that every number has two square roots.

$$z = a + ib$$
$$z^* = a - ib$$
$$z^* z = a^2 + b^2 = r^2$$
$$a = r\cos\theta$$
$$b = r\sin\theta$$

$$z = re^{i\theta} \Rightarrow \sqrt{z} = (re^{i\theta})^{1/2} = \sqrt{r}\, e^{i\theta/2}$$

$$(\sqrt{z})^* = \left(\sqrt{r}\, e^{i\theta/2}\right)^* = \sqrt{r}\, e^{-i\theta/2}$$

same as this

$$z^* = re^{-i\theta}$$

$$\sqrt{z^*} = (re^{-i\theta})^{1/2} = \sqrt{r}\, e^{-i\theta/2}$$